# Question 1

1 a) i)

↔

→

→

¬

¬

p

p

q

q

1 a) ii)

p → q is true iff p is false or q is true.

Hence, p → q is true iff ¬ p is true or ¬ q is false.

Hence, p → q is true iff ¬ q is false or ¬ p is true.

But ¬ q → ¬ p is true iff ¬ q is false or ¬ p is true.

Hence, p → q is true iff ¬ q → ¬ p is true.

1 a) iii)

p → q ↔ ¬ q → ¬ p

≡ (p → q) **∧** (¬ q → ¬ p) v ¬(p → q) **∧** ¬(¬ q → ¬ p) by A ↔ B ≡ A **∧** B v ¬A **∧** ¬B

≡ (¬p v q) **∧** (¬¬ q v ¬ p) v ¬(¬p v q) **∧** ¬(¬¬ q v ¬ p) by A → B ≡ ¬A v B

≡ (¬p v q) **∧** (q v ¬ p) v ¬(¬p v q) **∧** ¬(q v ¬ p) by ¬¬A ≡ A

≡ (¬p v q) **∧** (¬p v q) v ¬(¬p v q) **∧** ¬(¬p v q) by A v B ≡ B v A

≡ (¬p v q) v ¬(¬p v q) by A **∧** A ≡ A

≡ T by A v ¬A ≡ T

(Another way to do iii?????)

p → q ↔ ¬ q → ¬ p

= (¬p v q) ↔ (¬¬q v ¬ p) by A → B ≡ ¬A v B

= (¬p v q) ↔ (q v ¬ p) by ¬¬A ≡ A

= (¬p v q) ↔ (¬p v q) by commutativity of v

= ((¬p v q) **∧** (¬p v q)) v (¬(¬p v q) **∧** ¬(¬p v q)) by A ↔ B ≡ A **∧** B v ¬A **∧** ¬B

= (¬p v q) v ¬(¬p v q) idempotence of **∧**

= T by A v ¬A ≡ T

p → q ↔ ¬ q → ¬ p

= p → q ↔ ¬¬q v ¬ p by A → B ≡ ¬A v B

= p → q ↔ q v ¬ p by ¬¬A ≡ A

= p → q ↔ ¬p v q by commutativity of v

= ((p → q) →(p → q)) **∧** ((p → q) →(p → q)) by A ↔ B ≡ (A →B) **∧** (B → A)

= T **∧** ((p → q) →(p → q)) by A → A ≡ T

= T **∧** T by A → A ≡ T

= T by A **∧** T ≡ A

1 b)

(p → q) **∧** ¬q → ¬p

Proof in ND:

1 (p → q) **∧** ¬q (ass)

2 p (ass)

3 p → q **∧**E(1)

4 q →E(3,2)

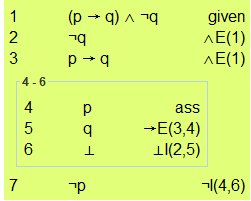
5 ¬q **∧**E(1)

6 **⊥ ⊥**I(4,5)

7 ¬p ¬I(2,6)

8 (p → q) **∧** ¬q → ¬p →I(1,7)

Pandora proof:



1 p △ (q △ r) (given)

2 ¬ r (ass)

3 p ∧ q (ass)

4 p ∧E(3)

5 q △ r (ass)

6 ¬ q △E(5,2)

7 q ∧E(3)

8 ⊥ ⊥I(6,7)

9 ¬(q △ r) ¬I(5,8)

10 ¬p △E(1,9)

11 ⊥ ⊥I(4,10)

12 ¬(p ∧ q) ¬I(3,11)

13 (p ∧ q) △ r △I(2,12)

1 c) i)

+

1 c) ii)

△ introduction:

1 ¬ B (ass)

…

2 ¬ A

3 ¬ B → ¬ A → I(1,2)

4 A (ass)

5 ¬ B (ass)

6 ¬ A → E(3,5)

7 ⊥ ⊥I(4,6)

8 B PC(5,7)

9 A → B → I(4,8)

Lines 1 to 2 are the same as in the definition of △I. Then instead of applying △I we a combination of → rules and PC to obtain A → B which is the same as A △B according to our assumption. So △I is a derived rule of regular ND collapsing lines 3 to 8 above.

△ elimination:

1 A → B

…

2 ¬ B

3 A (ass)

4 B → E(1,3)

5 ⊥ ⊥I(2,4)

6 ¬ A ¬I(3,5)

Lines 1 to 2 are the same as in the definition of △E. Then instead of applying △E we use a combination ¬I of →E ¬ A. So △E is a derived rule of regular ND collapsing lines 3 to 5 above.

Alternative answer for △ elimination?: according to part b, we can indeed derive from ND.

1 c) iii)

Soundness of ⊢\*:

If A\_1, A\_2, …, A\_n ⊢\* B then A\_1, A\_2, …, A\_n ⊨ B.

If A\_1, A\_2, …, A\_n ⊢\* B then A\_1, A\_2, …, A\_n ⊢ B in regular ND, as we can replace any occurrences of △I and △E in the proof regular ND rules as in ii) as they are derived rules of regular ND. But then A\_1, A\_2, …, A\_n ⊨ B because regular ND is sound.

Hence, ⊢\* is sound as well.

# Question 2

2 a) i) ¬∃x(M(x,f)) or ∀x¬(M(x,f))

2 a) ii) ∀x(P(x) → M(f,x))

2 a) iii) ∀x∀y∀z(P(x) **∧** M(x,y) **∧**  M(x,z) → y=z)

Alternative answer?: ∀x[P(x) →∀y∀z[M(x,y) **∧** M(x,z) → y=z]]

Another alternative(?): ∀x[P(x) →∃y∀z[M(x,z) → y=z]]

2 a) iv) ∀x(P(x) **∧** ∃y(M(x,y) **∧** P(y)) → M(f,x))

(simpler) Alternative:

2 a) v) ∀x∀y∀z(P(y) **∧** P(z) **∧** M(x,y) **∧**  M(x,z) → y=z)

Another answer ?:

∀y∀z(P(y) **∧** P(z) **∧** y ≠ z → ¬∃x[M(x,y) **∧**  M(x,z)])

∀x∀y∀z(y ≠ z **∧** P(y) **∧** P(z) **∧** M(x,y) → ¬M(x,z))

(Longer but still right?):

∀a∀b∀c∀d[P(a) **∧** P(b) ∧ M(c,a) ∧ M(d,b) ∧ a != b → c != d]

2 b) i) A and B are both true in C.

2 b) ii) Nobody manges 1 so its prediction is vacuously correct. So 1 gets a bonus.

Now 2 and 3 do not get bonus as they are managed by 1.

If 4 gets a bonus then this contradicts its prediction because it manages itself.

So 4 cannot get a bonus. Now 5 gets a bonus because it’s only manager 4 does not.

That make 4’s prediction false as 5 manages it, so it indeed does not get a bonus.

In conclusion, only 1 and 5 receive bonuses and the others do not.

2 c) ∀y(∃xM(x,y) → P(y))

≡ ∀y∀x(M(x,y) → P(y)) by ∃xA → B ≡ ∀x(A → B) when x is not free in B

≡ ∀x∀y(M(x,y) → P(y)) by ∀y∀xA ≡ ∀x∀yA

≡ ∀x∀y¬¬(M(x,y) → P(y)) by ¬¬A ≡ A

≡ ∀x¬∃y¬(M(x,y) → P(y)) by ∀x¬A ≡ ¬∃xA

≡ ∀x¬∃y¬(¬M(x,y) v P(y)) by A → B ≡ ¬A v B

≡ ∀x¬∃y(¬¬M(x,y) **∧** ¬P(y)) by ¬(A v B) ≡ ¬A **∧** ¬B

≡ ∀x¬∃y(M(x,y) **∧** ¬P(y))w by ¬¬A ≡ A

Slightly easier way?

∀y(∃xM(x,y) → P(y))

≡ ∀y∀x(M(x,y) → P(y)) by ∃xA → B ≡ ∀x(A → B) when x is not free in B

≡ ∀x∀y(M(x,y) → P(y)) by ∀y∀xA ≡ ∀x∀yA

≡ ∀x∀y¬(M(x,y) **∧** ¬P(y)) by A → B ≡ ¬(A **∧** ¬B)

// ≡ ∀x∀y(¬M(x,y) v P(y)) by A → B ≡ ¬A v B

// ≡ ∀x∀y¬(M(x,y) **∧** ¬P(y)) by ¬(A v B) ≡ ¬A **∧** ¬B

≡ ∀x¬∃y(M(x,y) **∧** ¬P(y)) by ∀x¬A ≡ ¬∃xA

2 d)

1 ∀xP(x) (given)

2 ∃x∀y(P(y) → y=x) (given)

3 ∀y(P(y) → y=c) ass

4 d ∀I const

5 P(d) ∀E(1)

6 d=c∀→E(3,5)

7 ∀y(y=c) ∀I(4,6)

8 ∃x∀y(y=x) ∃I(7)

9 ∃x∀y(y=x) ∃E(2,3,8)

